Journal of Statistical Physics, Vol. 77, Nos. 3/4, 1994

Percolation of Level Sets for Two-Dimensional Random Fields with Lattice Symmetry

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Received January 5, 1994; final April 20, 1994

Let $\psi(x)$, $x \in \mathbb{R}^2$, be a random field, which may be viewed as the potential of an incompressible flow for which the trajectories follow the level lines of ψ. Percolation methods are used to analyze the sizes of the connected components of level sets $\{x: \psi(x) = h\}$ and sets $\{x: \psi(x) \leqslant h\}$ in several classes of random fields with lattice symmetry. In typical cases there is a sharp transition at a critical value of h from exponential boundedness for such components to the existence of an unbounded component. In some examples, however, there is a nondegenerate interval of values of h where components are bounded but not exponentially so, and in other cases each level set may be a single infinite line which visits every region of the lattice.

KEY WORDS: Lagrangian trajectory; incompressible flow; turbulent diffusion; percolation; statistical topography; minimal spanning tree; random field; shot noise.

1. INTRODUCTION

Let $\mathbf{V}(x, \omega)$, $x \in \mathbb{R}^d$, be a homogeneous ergodic incompressible flow in dimension $d \geqslant 2$. Lagrangian trajectories are the trajectories of passive particles imbedded in the flow, that is, solutions of the ODE

$$\dot{x}_t = \mathbf{V}(x_t, \omega) \tag{1.1}$$

Here ω can be considered an index in the ensemble Ω of all realizations of the random flow $\mathbf{V}$; the probability space $(\Omega, \mathscr{F}, P)$ has probability measure P invariant and ergodic with respect to translations of $\mathbb{R}^d$. The

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0022-4715/94/1100-0627$07.00/0

problem of the qualitative structure of the solutions of (1.1) is of great physical importance, because the transport properties of the flow, such as heat propagation, depend on this structure. Time-independent random flows are especially important in plasma physics, where velocity fields are usually generated by magnetic fields which are themselves fixed over time, or changing only very slowly.

The physics literature on this subject is extensive (see refs. 1–7 and the references therein; ref. 2 is a review), but the analysis is generally based on numerical simulations and physical intuition. Refs. 8 and 9 are among the few mathematically rigorous studies of the problem.

The central working physical hypothesis about Lagrangian trajectories—sometimes called the *Sagdeev hypothesis*, after the Russian astrophysicist R. Sagdeev—can be formulated as follows: for $d=2$, if the mean $\langle \mathbf{V} \rangle = 0$, then in "typical situations" the Lagrangian trajectory containing a given point x_0 is a bounded loop P-a.s., and moreover, the size of this loop has finite moments of all orders, even perhaps an exponential moment. Transport of passive particles is then impossible. If the model incorporates a nonzero molecular diffusivity, i.e., a small diffusion term is added to (1.1), nontrivial transport will occur; in some rescaled limit there is turbulent diffusion. For $d \geqslant 3$, with again $\langle \mathbf{V} \rangle = 0$, in typical cases there is coexistence of two different types of Lagrangian trajectories: closed loops, which correspond to islands of stability for the system (1.1), and unbounded trajectories, which underlie the transport of passive particles for arbitrarily small molecular diffusivities. Thus the problem of describing Lagrangian trajectories is closely related to questions about turbulent diffusion.

In this paper we will consider some two-dimensional models which are hybrids of lattice and continuum models, where it is possible to get fairly complete information about the structure of the Lagrangian trajectories, using results and ideas from percolation theory. Related models were studied in refs. 1, 4, and 7. Besides proving results about these models, our goal is to formulate some new mathematical problems in the area.

2. RANDOM FIELDS WITH LATTICE SYMMETRY

For $d=2$ an arbitrary incompressible vector field $\mathbf{V}(x, \omega)$ with $\langle \mathbf{V} \rangle = 0$ can be represented as the curl of a homogeneous scalar potential $\psi(x)$, i.e.,

$$\mathbf{V} = (-\partial\psi/\partial x_2, \partial\psi/\partial x_1)$$

This means that each Lagrangian trajectory is an appropriately parametrized level line of ψ, at least provided there are no critical points of ψ (where $\nabla\psi = 0$) on the level line. Let

$$S_h = \{x: \psi(x) = h\}$$
$$S_{\leqslant h} = \{x: \psi(x) \leqslant h\}$$

and

$$\mathscr{H}_{\rm cr} = \{h \in \mathbb{R}: S_h \text{ contains a critical point of } \psi\}$$

If $\psi \in C^1$, then by Sard's Theorem, the set $\mathscr{H}_{\rm cr}$ has Lebesgue measure 0. If $x_0 \in S_h$ and $h \notin \mathscr{H}_{\rm cr}$, then the Lagrangian trajectory starting at x_0 will be periodic if and only if it is closed.

It is common to think of ψ as the elevation of a landscape and h as a level to which this landscape has been filled with water. Then $S_{\leqslant h}$ consists of lakes and/or an infinite ocean, and its complement consists of islands and/or an infinite land mass. The usual assumption in the physics literature (see, e.g., ref. 2) is that, for "typical" random fields (stationary, ergodic, rapidly decaying correlations, etc.) there is a sharp transition at some critical value of h from a land mass with lakes to an ocean with islands.

One reasonably physically realistic type of potential is shot noise, that is, a random field of the form

$$\psi(x) = \sum_i A_i \varphi(|x - x_i|)$$

where $\{A_i, i \geqslant 1\}$ are i.i.d. random variables with $\langle |A_i| \rangle < \infty$, $X = \{x_i: i \geqslant 1\}$ is the set of sites of a lattice, a Poisson process, or some other locally finite stationary point process, and φ: $\mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\int_{\mathbb{R}^2} \varphi(|x|)\, dx < \infty$$

Analysis of the level lines for such potentials is in general a complex problem, even if φ and/or the correlation

$$B(x) = \langle \psi(0)\, \psi(x) \rangle - \langle \psi(0) \rangle \langle \psi(x) \rangle$$

decrease rapidly with increasing $|x|$. In the Gaussian case it is known[9] that for some h_0, for $|h| > h_0$ the level set S_h has only bounded connected components, and the lengths of these components have an exponential moment. Let us say that a subset of $\mathbb{R}^2$ *percolates* if it has an unbounded connected component; the corresponding percolation problem for $|h| < h_0$ is not solved.

Nonetheless, by taking the set X of sites to be a lattice, we can consider a class of continuous models with lattice symmetry, to which results on two-dimensional lattice percolation can be applied; see ref. 10 or 11 for the necessary background on lattice percolation. Other lattice-symmetric models, not only of shot noise form, have been considered in refs. 1, 3, 4, and 7.

Let $\mathbb{L}$ be one of three standard two-dimensional lattices: the square lattice $\mathbb{Z}^2$, the triangular lattice $\mathbb{T}^2$, and the hexagonal lattice $\mathbb{H}^2$, each with nearest-neighbor bonds of length 1. The lattice $\mathbb{L}$ divides the plane into *faces* which are either squares, triangles, or hexagons with sides of unit length. Let D denote the closed unit disk in $\mathbb{R}^2$ and suppose φ: $[0, \infty) \to [0, 1]$ is a smooth function strictly decreasing on its support $[0, 1/2 + \varepsilon]$, with $0 < \varepsilon < 1/2$ chosen small enough so that the translates $x + (1/2 + \varepsilon) D$, $x \in \mathbb{L}$, intersect only pairwise. Suppose also that $\varphi(0) = 1$ and that φ is strictly convex on $[1/2 - \varepsilon, 1/2 + \varepsilon]$. The latter ensures that if b is a bond of $\mathbb{L}$ with endpoints x and y, then

$$\inf_{w \in b} (\varphi(|w - x|) + \varphi(|w - y|)) = \inf_{t \in [0,1]} (\varphi(t) + \varphi(1 - t)) = 2\varphi(1/2)$$

or more generally that for arbitrary positive constants c_1 and c_2,

$$\inf_{w \in b} (c_1 \varphi(|w - x|) + c_2 \varphi(|w - y|)) = \inf_{t \in [0,1]} (c_1 \varphi(t) + c_2 \varphi(1 - t))$$

is uniquely achieved at $w = t_0 y + (1 - t_0) x$ for some $t_0 = t_0(c_1, c_2, \varphi) \in [1/2 - \varepsilon, 1/2 + \varepsilon]$.

Let H denote the set of points in $\mathbb{R}^2$ for which 0 is the closest site in $\mathbb{L}$: $H = [-1/2, 1/2]^2$ if $\mathbb{L} = \mathbb{Z}^2$, H is a hexagon centered at 0 if $\mathbb{L} = \mathbb{T}^2$, and H is a triangle centered at 0 if $\mathbb{L} = \mathbb{H}^2$. Let U be a random variable uniform over H, let $X = \{x_i, i \geq 1\}$ be the set of sites of the lattice $\mathbb{L}$, and let $\{A_i, i \geq 1\}$ be i.i.d. symmetric random variables. Our random potential is then

$$\psi(x) = \sum_i A_i \varphi(|x - x_i - U|)$$

It is clear that ψ is invariant and ergodic with respect to translations of $\mathbb{R}^2$. Further,

$$B(x) = 0 \qquad \text{for} \quad |x| > 1 + 2\varepsilon$$

Of course due to the lattice structure the tail σ-field of ψ is not trivial. Let Γ denote the connected component of the level set $S_{\psi(0)}$ which contains the origin, and

$$R = \sup\{ |x| : x \in \Gamma\}$$

Model 1. We first consider the triangular lattice and $A_i = \pm 1$ with probability 1/2 each. Here we observe a sort of critical interval of levels, $-h_{cr} < h < h_{cr}$, where components of the level sets S_h and regions $S_{\leqslant h}$ are bounded but not with an exponential moment. Thus, in contrast with the usual assumption made in the physics literature, there is no sharp transition at some critical value of h from an unbounded land mass with exponentially bounded lakes to an unbounded ocean with exponentially bounded islands. No level lines percolate.

Theorem 2.1. For the lattice $\mathbb{L} = \mathbb{T}^2$ and $A_i = \pm 1$ with probability 1/2 each, and for $h_{cr} = 2\varphi(1/2)$,

(a) $S_{\leqslant h}$ percolates if and only if $h \geqslant h_{cr}$

(b) S_h percolates for no h

(c) for some positive constants c_3 and γ, $P(R > t) \geqslant c_3 t^{-\gamma}$

(d) for $|h| > h_{cr}$, the connected components of S_h have diameter bounded by $1 + 2\varepsilon$

(e) for all $|h| < h_{cr}$, for some positive constants $c_4 = c_4(h)$ and α,

$$P(R > t \mid \psi(0) = h) \geqslant c_4 t^{-\alpha}$$

The proof will require the following result.

Lemma 2.2. Let $p_n \geqslant 0$ and $r_n = \sum_{k=n}^{\infty} p_k$. Suppose that for some positive constants a_1, a_2, α, and β,

$$a_1 n^{-\beta} \leqslant r_n \leqslant a_2 n^{-\alpha} \qquad \text{for all} \quad n \geqslant 1$$

Let $\gamma > 0$ and $\delta = \beta(\gamma + \alpha)/\alpha$; then there exists $a_3 > 0$ such that

$$\sum_{k=n}^{\infty} k^{-\gamma} p_k \geqslant a_3 n^{-\delta} \qquad \text{for all} \quad n \geqslant 1$$

Proof. Choose c so that $2a_2 c^{-\alpha} < a_1$. If $k \geqslant c n^{\beta/\alpha}$, then

$$r_k \leqslant a_2 k^{-\alpha} \leqslant a_2 c^{-\alpha} n^{-\beta} \leqslant r_n/2$$

Hence from summation by parts,

$$\sum_{k=n}^{\infty} k^{-\gamma} p_k = n^{-\gamma} r_n - \sum_{k=n+1}^{\infty} r_k[(k-1)^{-\gamma} - k^{-\gamma}]$$

$$= \sum_{k=n+1}^{\infty} (r_n - r_k)[(k-1)^{-\gamma} - k^{-\gamma}]$$

$$\geqslant (r_n/2) \sum_{k \geqslant cn^{\beta/\alpha}} [(k-1)^{-\gamma} - k^{-\gamma}]$$

$$\geqslant (r_n/2)(cn^{\beta/\alpha})^{-\gamma}$$

$$\geqslant a_3 n^{-\delta} \quad \blacksquare$$

In an abuse of terminology, when confusion is unlikely, we will identify a planar graph with the set of its bonds, and identify a bond with the line segment in $\mathbb{R}^2$ connecting its endpoints.

Proof of Theorem 2.1. We need some facts about site percolation on $\mathbb{T}^2$ with sites independently occupied with probability p. At the critical probability $p_c = 1/2$, neither occupied nor vacant sites percolate.[12] Therefore every site is surrounded by a sequence of disjoint circuits C_i, $i \geqslant 1$, with all sites in C_i occupied if i is even, and all sites vacant if i is odd. In the present model we can designate a site x_i as a plus, or occupied, site if $A_i = 1$, and a minus, or vacant, site if $A_i = -1$. The circuits C_i can then be labeled *plus* and *minus circuits*.

If $h < h_{\text{cr}}$ then $S_{\leqslant h}$ cannot cross any plus circuit, so does not percolate. If $h \geqslant h_{\text{cr}}$, then each component of the complement of $S_{\leqslant h}$ is contained in the support of a single function $\varphi(|\cdot - x_i - U|)$, i.e., $S_{\leqslant h}$ crosses every bond, so it is clear that $S_{\leqslant h}$ percolates, and (a) is proved. Statement (d) is proved similarly. Statement (b) follows from the fact that no level line can cross both plus and minus circuits.

Fix h, $0 < |h| < h_{\text{cr}}$; let

$$B_i = \{x \in \mathbb{R}^2 \colon |x - x_i| \leqslant 1/2 + \varepsilon\}$$

and set

$$V = \{x \in \mathbb{R}^2 \colon x \in B_i \text{ for exactly one } i\}$$

For $x \in \mathbb{R}^2$ let $X(x)$ be a site x_i which minimizes $|x - x_i|$. Since the regions B_i intersect only pairwise, if x_i is a plus site, then $(S_h - U) \cap V \cap B_i$ consists of six equal nonempty arcs of a circle, which we call *basic arcs* of $S_h - U$, and

$$p_V(h) = P(-U \in V \mid \psi(0) = h) > 0$$

Here $S_h - U$ denotes the translate of the set S_h by $-U$.

One can place a bond between every nearest-neighbor pair of sites which have the same sign; the resulting connected components are called *plus clusters* and *minus clusters*. The *outer boundary* of a cluster C is

$$\partial C = \{x \in C\colon x \text{ is connected to infinity by a lattice path outside } C \backslash \{x\}\}$$

Let $|F|$ denote the number of sites in a set F. If x_i is a site in $C\backslash\partial C$, then the hexagon $\{x\colon X(x) = x_i\}$ of area $\sqrt{3/4}$ is entirely inside the contour ∂C; it follows that

$$\begin{aligned} |\partial C| &= \text{length of } \partial C \\ &\geqslant 2\sqrt{\pi}\cdot(\text{area enclosed by } \partial C)^{1/2} \\ &\geqslant c_5(|C| - |\partial C|)^{1/2} \end{aligned}$$

for some constant c_5, which implies

$$|\partial C| \geqslant c_6\,|C|^{1/2} \tag{2.1}$$

Let C_0 denote the cluster of 0 [necessarily a plus cluster if $\psi(0) > 0$ and a minus cluster if $\psi(0) < 0$]. Now S_h includes a loop γ_0 which encloses C_0, and hence encloses area at least $c_7\,|C_0|$, for some $c_7 = c_7(h)$. Therefore the length of γ_0 is at least $c_8\,|C_0|^{1/2}$. If $-U \in \gamma_0$, then $\Gamma = \gamma_0$, so $R \geqslant c_8\,|C_0|^{1/2}$. If $0 = x_i \in \partial C_0$, then at least two of the six basic arcs in B_i are part of γ_0. It follows that

$$\begin{aligned} P(R \geqslant c_8 n^{1/2} \mid \psi(0) = h) &\geqslant P(-U \in \gamma_0,\, |C_0| \geqslant n \mid \psi(0) = h) \\ &\geqslant P(-U \in V,\, 0 \in \partial C_0,\, |C_0| \geqslant n \mid \psi(0) = h)/3 \\ &= P(0 \in \partial C_0,\, |C_0| \geqslant n)\, p_V(h)/3 \\ &\geqslant \tfrac{1}{3} p_V(h) \sum_{k=n}^{\infty} c_6 k^{-1/2} P(|C_0| = k) \end{aligned} \tag{2.2}$$

The equality in (2.2) follows from the fact that knowing $-U \in V$ and $\psi(0) = h$ does not condition the sign (plus or minus) of any site other than $X(-U)$. The last inequality follows from (2.1).

Now for some constants a_1, a_2, and $\alpha > 0$,

$$a_1 n^{-1/2} \leqslant P(|C_0| \geqslant n) \leqslant a_2 n^{-\alpha}$$

(see ref. 10, Theorem 9.89; the proof for site percolation on the triangular lattice is similar.) Hence for $0 < |h| < h_{cr}$, (e) follows from Lemma 2.2 and (2.2), and then (c) follows.

It remains to prove (e) for $h=0$. Let $\{J_i, i \geq 1\}$ be a listing of the faces, and $K_i = J_i \backslash \bigcup_j B_j$. For any bond b separating two adjacent faces J_i and J_k, the dual bond b^* is the perpendicular bisector of b with endpoints at the centers of J_i and J_k. Let S denote the union of all bonds b^* dual to bonds b for which the endpoints of b are of opposite sign. Then

$$S_0 = S \cup \bigcup_{i \geq 1} K_i \tag{2.3}$$

Given $\psi(0)=0$, we thus have $-U \in \bigcup_i K_i$ a.s., so we can define K to be the region K_i which contains $-U$. If $0 \in \partial C_0$ and K is connected to infinity outside C_0, then Γ contains a curve surrounding C_0. At least two of the six triangular faces J_i with 0 as one vertex are connected to infinity outside C_0, so for some constant c_9,

$$\begin{aligned} P(R \geq t \mid \psi(0)=0) &\geq P(\operatorname{diam}(C_0) \geq t, 0 \in \partial C_0 \mid \psi(0)=0)/3 \\ &\geq P(|C_0| \geq c_9 t^2, 0 \in \partial C_0)/3 \end{aligned}$$

and (e) follows as for $0 < |h| < h_{\text{cr}}$. ■

Model 2. We consider the square or hexagonal lattice and $A_i = \pm 1$ with probability 1/2 each. In contrast to model 1, here there is no critical interval, but rather a single critical level of 0, which is the only level which percolates. All other level lines are exponentially bounded.

Theorem 2.3. For the lattice $\mathbb{L} = \mathbb{Z}^2$ or $\mathbb{H}^2$ and $A_i = \pm 1$ with probability 1/2 each,

(a) $S_{\leq h}$ percolates if and only if $h \geq 0$

(b) S_h percolates only for $h=0$

(c) for $|h| > 2\varphi(1/2)$ the connected components of S_h have diameter bounded by $1+2\varepsilon$

(d) for some constants c_i, for all $h \neq 0$, $P(R > t \mid \psi(0)=h) \leq c_{10} \exp(-c_{11} t)$

This model thus has the main properties considered "generic" in the physics literature: bounded S_h for all h except the critical level $h=0$, with an exponential moment for $h \neq 0$.

Proof of Theorem 2.3. We continue with the terminology of the proof of Theorem 2.1. Any pair of sites from the boundary of a single face (or the line segment connecting these two sites) is called a **-bond*, and a **-circuit* is a circuit composed of *-bonds. A circuit is a *plus *-circuit* or a *minus *-circuit* if all sites on it have the corresponding sign. A **-path* is a path consisting of *-bonds, and we say there is **-percolation* of plus (or minus) sites if there is an infinite *-path on which all sites are plus (or minus) sites. A site with a given sign is part of a finite cluster if and only if it is surrounded by a *-circuit of opposite sign.

The term circuit or cluster, without the *, still refers to connection via nearest-neighbor bonds.

For nearest-neighbor site percolation on $\mathbb{Z}^2$ or $\mathbb{H}^2$, we have[11,13]

$$p_c > 1/2 \tag{2.4}$$

so at density 1/2, neither plus nor minus sites percolate. Therefore every site is surrounded by a disjoint sequence of *-circuits C_i, $i \geqslant 1$, with all sites in C_i plus sites if i is even, and all sites minus if i is odd. Given a plus *-circuit, there is a curve passing through the same sites and faces as the plus *-circuit, in the same order, such that $\psi(x) \geqslant 0$ at every x in the curve; one merely moves the plus *-circuit slightly so it does not pass through B_j for any minus site x_j. This curve cannot be crossed by $S_{\leqslant h}$ for $h < 0$, so S_h and $S_{\leqslant h}$ do not percolate for $h < 0$. The same proof applies to S_h for $h > 0$, so to prove (a) and (b) it remains to show S_0 percolates.

It follows from (2.3) that every bounded component of S_0 is surrounded by either a plus circuit or a minus circuit, as this is the only kind of circuit that is not crossed by S_0. From ref. 12, or ref. 11, Corollary 3.1, we have that the critical probability p_c^* for *-percolation satisfies

$$p_c^* = 1 - p_c$$

Therefore by (2.4), *-percolation of minus sites occurs at density 1/2, so there can be only finitely many plus circuits surrounding the origin. Similarly, only finitely many minus circuits surround the origin, so S_0 must have an unbounded component, i.e., S_0 percolates. This completes the proof of (a) and (b).

If $\psi(0) = h \neq 0$, then Γ is contained in $\bigcup_{i: x_i \in C_0} B_i$, so

$$P(R > t \mid \psi(0) = h) \leqslant P(\text{radius}(C_0) > t - 2)$$

Since $1/2 < p_c$, the latter decays exponentially[14-16] in t, which proves (d). Statement (c) is obvious. ■

Model 3. We consider the triangular lattice with general symmetric A_i. Here a critical interval as in model 1 exists if and only if $0 \notin \text{supp}(A_i)$.

Theorem 2.4. For the lattice $\mathbb{L} = \mathbb{T}^2$, A_i symmetric, $h_0 = \inf \sup(|A_i|)$, and $h_{\text{cr}} = 2h_0 = \varphi(1/2)$,

(a) if $h_0 > 0$ and $P(A_i = h_0) = 0$, then

  (i) $S_{\leqslant h}$ percolates if and only if $h > h_{cr}$

  (ii) S_h percolates for no h

(b) if $h_0 = 0$ and $P(A_i = 0) = 0$, then

  (i) $S_{\leqslant h}$ percolates if and only if $h > 0$

  (ii) S_h percolates for no h

  (iii) for some constants c_i, for all $h \neq 0$, $P(R > t \mid \psi(0) = h) \leqslant c_{12} \exp(-c_{13} t)$

  (iv) for some constants c_{14} and α, $P(R > t \mid \psi(0) = 0) \geqslant c_{14} t^{-\alpha}$

(c) if $h_0 > 0$ and $P(A_i = h_0) > 0$, then

  (i) $S_{\leqslant h}$ percolates if and only if $h \geqslant h_{cr}$

  (ii) S_h percolates for no h

(d) if $h_0 = 0$ and $P(A_i = 0) > 0$, then

  (i) $S_{\leqslant h}$ percolates if and only if $h \geqslant 0$

  (ii) S_h percolates only for $h = 0$

Further, if $h_0 > 0$, then (c) and (e) of Theorem 2.1 are valid.

The proof of Theorem 2.4 will require the following result.

Lemma 2.5. For the nearest-neighbor graph G on the lattice $\mathbb{T}^2$, let $\mathscr{B}$ be the set of bonds, and let p, $\varepsilon > 0$. Suppose that each site is independently labeled plus with probability p, minus with probability $1 - p$, and, independently, marked with probability ε, unmarked with probability $1 - \varepsilon$. Let

$$A = \{b \in \mathscr{B}\text{: at least one endpoint of } b \text{ is a minus site}\}$$

$$B = \{b \in \mathscr{B}\text{: both endpoints of } b \text{ are marked sites}\}$$

$$Z = \{x \in \mathbb{T}^2\text{: } x \leftrightarrow 0 \text{ by a path of bonds } b \notin A \cup B\}$$

$$W = \{b^*\text{: } b \in A \cup B\}$$

Then for $p = 1/2$, W percolates, and for some constants c_i

$$P(\operatorname{diam}(Z) > n) \leqslant c_{14} \exp(-c_{15} n) \tag{2.5}$$

Note the conclusion that W percolates is false for $\varepsilon = 0$, for then W cannot cross any plus circuit, and every site is surrounded by a plus circuit a.s. One may think of an adjacent pair of marked sites as creating a breach in a plus circuit where W can cross it. Percolation of W says that any positive ε is enough to create such breaches. Since $p_c = 1/2$, this would be trivial if only a single marked site were needed for a breach, but the need for an adjacent pair complicates things.

Proof of Lemma 2.5. Let $\xi_1 = (1, 0)$, $\xi_2 = (1/2, \sqrt{3}/2)$, so that the points $z_{ij} = i\xi_1 + j\xi_2$, $i, j \in \mathbb{Z}$, are the sites of the triangular lattice. We wish to compare site percolation on G with site percolation on a modified graph $\tilde{G}$, constructed by adding a site y_{ij} at the midpoint of the bond $\{z_{i,j+1}, z_{i+1,j}\}$ for each $i, j \in \mathbb{Z}$ with i even. The matching graph $\tilde{G}^*$ is then obtained from $\tilde{G}$ by adding a bond from z_{ij} to y_{ij}, a bond from y_{ij} to $z_{i+1,j+1}$, and a bond from $z_{i,j+1}$ to $z_{i+1,j}$, all for each $i, j \in \mathbb{Z}$ with i even. (Note $\tilde{G}^*$ is not planar.) We now declare a site to be *vacant* if either it is a plus site in $\mathbb{T}^2$ or it is a site y_{ij} and both $z_{i,j+1}$ and $z_{i+1,j}$ are marked. Otherwise the site is called *occupied*. Because the additional sites y_{ij} exist only for even i, the occupied/vacant properties for distinct sites are independent.

By Theorem 1 of ref. 17, for fixed $\varepsilon > 0$ the critical value $p_c(\tilde{G})$ of p for percolation of occupied sites in $\tilde{G}$ is strictly less than the critical value 1/2 for G, so that also the critical value[12] $p_c(\tilde{G}^*) = 1 - p_c(\tilde{G})$ of p for occupied percolation in $\tilde{G}^*$ is strictly greater than 1/2. Vacant sites percolate in $\tilde{G}$ only when $p > p_c(\tilde{G}^*)$ (ref. 10, Corollary 3.1). Thus for $p = 1/2$, occupied sites percolate in $\tilde{G}^*$, and (see Theorem 5.1 of ref. 11) the vacant cluster diameter in $\tilde{G}$ has exponential tails,[14,15,16] that is, (2.5) holds.

Let H_{ij} denote the hexagonal region bounded by $\{b^*: b$ is a bond of G with z_{ij} as one endpoint$\}$. Let $\tilde{\gamma}$ be an infinite path in $\tilde{G}^*$ in which all sites are occupied. It is easy to see that $\tilde{\gamma}$ can be chosen with the properties that

$$\text{any two sites in } \tilde{\gamma} \text{ which are adjacent in } \tilde{G} \text{ are adjacent in } \tilde{\gamma} \tag{2.6}$$

and

$$\text{if } y_{ij} \text{ is a site in } \tilde{\gamma}\text{, then the two adjacent sites of } \tilde{\gamma} \text{ are } z_{ij} \text{ and } z_{i+1,j+1} \tag{2.7}$$

for if (2.7) fails, then y_{ij} can be skipped over. Let γ be the corresponding infinite path in $\mathbb{R}^2$ obtained by replacing each bond of $\tilde{\gamma}$ with a line segment having the same endpoints. Then by (2.7), γ is contained in $\{b^*: b \in B\} \cup \bigcup_{z_{ij} \in \gamma} H_{ij}$. By (2.6) the latter set a.s. has an unbounded connected boundary, and that boundary is part of W. The percolation of W follows. ∎

Proof of Theorem 2.4. We can express A_i as $\varepsilon_i |A_i|$ with $\varepsilon_i = \pm 1$ with probability 1/2 each, independent of $|A_i|$. We now call a site x_i a plus site if $\varepsilon_i = 1$ and a minus site if $\varepsilon_i = -1$. As in the proof of Theorem 2.1, every site is surrounded by a disjoint sequence of circuits C_i, $i \geqslant 1$, with all sites in C_i plus sites if i is even and all sites minus sites if i is odd.

If $h < h_{cr}$, then $S_{\leqslant h}$ cannot cross any plus circuit, so does not percolate. If $P(A_j = h_0) = 0$, then this applies to $h = h_{cr}$ as well. If $h > h_{cr}$, let $s = h/2\varphi(1/2)$, and call a site x_j *marked* if $|A_j| \leqslant s$. Of course being marked is independent of being a plus or minus site. Note that $S_{\leqslant h}$ crosses all bonds b in the sets A and B of Lemma 2.5, i.e., $S_{\leqslant h}$ contains the set W of that lemma. Thus for $h > h_{cr}$, $S_{\leqslant h}$ percolates a.s. If $P(A_j = h_0) > 0$, then this applies to $h = h_{cr}$ as well. This proves (i) of (a)–(d).

If $h \neq 0$, then S_h cannot cross both plus and minus circuits. If $P(A_j = 0) = 0$, then this applies also to $h = 0$. This completes the proof of (ii) of (a)–(c). Suppose $P(A_n = 0) > 0$. One can place a bond between every nearest-neighbor pair of sites x_i, x_j for which both $A_i \leqslant 0$ and $A_j \leqslant 0$. Since the critical probability for $\mathbb{T}^2$ is 1/2, the resulting graph has a unique infinite cluster, which we denote $C^{\infty}_{\leqslant 0}$. We can analogously define $C^{\infty}_{\geqslant 0}$, and let γ be an infinite self-avoiding lattice path in $C^{\infty}_{\geqslant 0}$. For each k we let

$$D_k := \{x_i \in \mathbb{L}\colon x_i \leftrightarrow x_k \text{ by a lattice path entirely outside } C^{\infty}_{\leqslant 0}\}$$

D_k is the set of sites in the hole in $C^{\infty}_{\leqslant 0}$ which contains x_k, if $A_k > 0$. Then D_k is finite for all k, since $C^{\infty}_{\leqslant 0}$ includes a circuit around every site, a.s. Let

$$E_k := \bigcup_{i\colon x_i \in D_k} B_i$$

$$H_k := \{x \in \mathbb{R}^2\colon \text{there is no path from } x \text{ to infinity in } S_{\leqslant 0} \cup E_k^c\}$$

so $D_k \subset H_k$. Note that if $x \in E_k \backslash H_k$, then x is near the outer boundary of E_k in the sense that $x \in B_i \cap B_j$ for some $x_i \in D_k$ and $x_j \notin D_k$. Further, H_k is bounded and simply connected, $\psi = 0$ on ∂H_k, and any two sets H_k and H_m either coincide or are disjoint. Therefore the boundary of the set

$$\gamma \cup \Bigl(\bigcup_{k\colon x_k \in \gamma} H_k \Bigr)$$

is an unbounded connected subset of S_0, which completes the proof of (d)(ii).

If $\psi(0) = h > 0$, then, for the Z of Lemma 2.5, Γ is contained in $\bigcup_{i\colon x_i \in Z} B_i$, so

$$P(R > t \mid \psi(0) = h) \leqslant P(\text{diam}(Z) > t - 2)$$

The case of $\psi(0) = h < 0$ is similar. Thus (b)(iii) follows from Lemma 2.5.

The proof of (b)(iv) is the same as the proof of Theorem 2.1(e) for $h = 0$. The proofs of statements (c) and (e) of Theorem 2.1 remain the same. ■

3. INFINITE SPANNING TREES AND ASSOCIATED RANDOM FIELDS

For a finite set $X \in \mathbb{R}^2$, a (Euclidean) *minimal spanning tree* (MST) of X is a tree with site (i.e., vertex) set X and minimal total length of all bonds (i.e., edges.) More generally, given a finite graph G with site set X and bond set $\mathscr{B}$, and a function $f: \mathscr{B} \to [0, \infty)$, an *f-minimal spanning tree* (f-MST) of X is a tree with site set X and $\sum_{b \in \mathscr{B}} f(b)$ minimal among all such trees.

We will describe how a closely related infinite spanning tree T can be created for certain countably infinite sets $X \subset \mathbb{R}^2$ and functions f. It will be shown to have the property that for each site there is a unique bond of T emanating from that site through which the site is connected to infinity in T. In other words, removing any one bond of T leaves one finite and one infinite component. For "nice" X, for h near 0, each level set S_h of the potential

$$\psi(x) = d(x, T)$$

then includes a single infinite line; this line traces around the tree, missing at most isolated pockets. Here $d(\cdot, \cdot)$ denotes Euclidean distance, and $d(x, T) = \inf\{d(x, y): y \in T\}$.

In this context, the most natural example is to take X to be the set of sites of a Poisson process, take G to be the complete graph or the Delaunay triangulation (see ref. 5) on X, and take f to be Euclidean distance. However, technicalities are reduced somewhat, without altering the basic argument, if we take $X = \mathbb{L}$, one of the three standard two-dimensional lattices of Section 2, $\mathscr{B}$ the set of nearest-neighbor bonds, and $\{f(b), b \in \mathscr{B}\}$ i.i.d. random variables uniform in $[0, 1]$. Since the theme of this paper is lattice-based examples, we will take the latter course. More general results (with more complicated proofs) covering both the Poisson/Euclidean case and the lattice/i.i.d.-uniform case in all dimensions appear in ref. 18. Our techniques here are strongly two-dimensional.

Given a graph G with site set X and bond set $\mathscr{B}$, and given a finite $A \subset X$, $x \in A$, $f: \mathscr{B} \to [0, \infty)$ and $r > 0$, let G_A denote the graph with site set A and bond set $\{b = \{x, y\} \in G: x, y \in A\}$, and

$$Y_{<r}(x, A) = \{y \in A: x \leftrightarrow y \text{ by a path in } G_A \text{ consisting only of bonds } b \text{ with } f(b) < r\}$$

and

$$Y_{<r}(x) = Y_{<r}(x, X)$$

It is well known[19] that for finite A, if f is one-to-one,

$$\text{the } f\text{-MST of } A \text{ is } \{b = \{x, y\} \in G_A: y \notin Y_{<f(b)}(x, A)\} \tag{3.1}$$

This motivates us to define the *infinite f-MST*

$$T = \{b = \{x, y\} \in G: y \notin Y_{<f(b)}(x)\}$$

Let $\Lambda_n \uparrow X$ and let $T(n)$ denote the f-MST of Λ_n; from the above discussion, if $m < n$, $b = \{x, y\} \subset [-m, m]^2$, and $b \in T(n)$, then $b \in T(m)$. This means that there is a limiting set of bonds—those which remain in $T(n)$ as $n \to \infty$—which does not depend on $\{\Lambda_n\}$ and is precisely T.

A nearly identical structure was investigated by Aldous and Steele[20] for X the set of sites of a stationary point process and $f(b)$ the Euclidean length $|b|$, the difference being that in their analog of T, bonds $b = \{x, y\}$ with $Y_{<|b|}(x)$ and $Y_{<|b|}(y)$ disjoint and both infinite were excluded. However, as they point out, for the Poisson process the two structures are a.s. the same; see ref. 18.

Suppose f is one-to-one and the vertices of G have finite degree. It is clear that T is acyclic, because any cycle in G has a unique bond b maximizing f, which is necessarily not in T, by (3.1). Further, T has no finite components, for the f-minimizing bond connecting such a component to an outside vertex would necessarily be in T, again by (3.1). In particular, there are no one-point components, so T spans the vertex set. What we need to prove are the facts, conjectured by Aldous and Steele[20] in the case of stationary X and Euclidean f, that (i) T is connected, i.e., T is a tree, and (ii) T has *finite branches*, i.e., if any one bond is removed from T, exactly one of the two resulting components is infinite.

We turn now to G a two-dimensional lattice $\mathbb{L}$—square, triangular, or hexagonal—with nearest-neighbor bonds and i.i.d. uniform values of f. Let G^* denote the dual graph of G; each bond b in G has a unique dual bond b^* which is its perpendicular bisector.

Theorem 3.1. For the nearest-neighbor graph G on each of the three two-dimensional lattices $\mathbb{L} = \mathbb{Z}^2$, $\mathbb{T}^2$, or $\mathbb{H}^2$, and $\{f(b), b \in \mathscr{B}\}$ i.i.d. random variables uniform in $[0, 1]$, the infinite f-MST T is a.s. a tree with finite branches which spans $\mathbb{L}$.

A different method of constructing a stationary random tree, with finite branches, which spans $\mathbb{Z}^2$ was considered by Pemantle.[21]

Proof of Theorem 3.1. The fact that T spans $\mathbb{L}$ and is a.s. acyclic with no finite components was noted above. Let C be a connected component of T and suppose $C \neq T$. Let

$$\partial C = \{b = \{y, z\} \in \mathscr{B}: y \in C, z \notin C\}$$

and suppose $b_0 = \{y_0, z_0\} \in \partial C$. Then $b_0 \notin T$, so by (3.1) there exists a path γ_0 in G from y_0 to z_0 consisting entirely of bonds b with $f(b) < f(b_0)$. Let $b_1 = \{y_1, z_1\}$ be the first bond in γ_0 which is in ∂C—such a bond necessarily exists since $y_0 \in C$ and $z_0 \notin C$. Similarly, there exists a path γ_1 in G from y_1 to z_1 consisting entirely of bonds b with $f(b) < f(b_1)$, and a first bond $b_2 = \{y_2, z_2\}$ in γ_1 which is in ∂C; inductively, this process can be continued indefinitely. Let γ be the path in C which follows γ_0 from y_0 to y_1, then γ_1 from y_1 to y_2, and so on. Now the bonds b_i are distinct, since $f(b_0) > f(b_1) > \dots$, so infinitely many of the vertices y_i are distinct, so γ is an infinite path consisting entirely of bonds b with $f(b) < f(b_0)$.

Let p_c denote the critical probability for Bernoulli bond percolation on the lattice $\mathbb{L}$. It follows from the above that $f(b_0) \geqslant p_c$, and since b_0 is arbitrary, that $f(b) \geqslant p_c$ for all $b \in \partial C$. Since all connected components of T are infinite, so are all components of the dual boundary $\{b^*: b \in \partial C\}$. Since $C \neq T$, this dual boundary is nonempty. Thus $\{b^*: f(b) \geqslant p_c\}$ percolates. But for Bernoulli bond precolation on two-dimensional lattices, at the critical point there is a.s. no percolation of either bonds or dual bonds.[12] Thus the probability that there is a component $C \neq T$ is zero, i.e., T is connected a.s.

The existence of a bond b such that removing b from T leaves two infinite components is equivalent to the existence of a doubly infinite path $\dots \to v_{-1} \to v_0 \to v_1 \to \dots$ of distinct sites in T. Since there is no percolation at p_c, with probability one not all bonds b in such a path can have $f(b) \leqslant p_c$. Thus it suffices to consider a bond $b = \{x, y\}$ with $f(b) > p_c$. Let T_x and T_y be the two components of T remaining after b is removed, with $x \in T_x$ and $y \in T_y$. Suppose $b' = \{x', y'\} \in \partial T_x$, with $x' \in T_x$; necessarily $y' \in T_y$. There exist paths γ_x from x to x' in T_x and γ_y from y' to y in T_y. In any circuit the bond e with $f(e)$ maximal is necessarily not in T, by (3.1); in the circuit consisting of γ_x, b', γ_y, and b, the only bond not in T is b', so b' must maximize f in this circuit. In particular, $f(b') > f(b) > p_c$; this is valid for all $b' \in \partial T_x$. Therefore $\{e^*: e \in \partial T_x\} \subset \{e^*: f(e) > p_c\}$; as mentioned above, the latter set does not percolate. Since T_x and T_y are connected, so is $\{e^*: e \in \partial T_x\}$. Hence $\{e^*: e \in \partial T_x\}$ is finite, so either T_x or T_y is finite. ■

Formally, the random field $\psi(x) = d(x, T)$ is not homogeneous, as it is only invariant under shifts by an element of the lattice $\mathbb{L}$. This can be remedied by translating the entire configuration by the random variable U as in Section 2; to avoid unnecessary technicalities we will not do so here.

Corollary 3.2. Let T denote the f-MST of Theorem 3.1, with $\mathbb{L}$ the square lattice, and define the potential $\psi(x) = d(x, T)$, $x \in \mathbb{R}^2$. For each

$0 < h < 1/2$, the level line S_h is a.s. a single infinite line which enters every square of the lattice. Further, $\mathrm{cov}(\psi(s+x), \psi(s)) \to 0$ uniformly in s as $|x| \to \infty$.

Proof. The description of S_h is immediate from Theorem 3.1. For each nearest-neighbor pair $\{x, y\}$ in the lattice, let $\Lambda_n(x, y) = \{(x+y)/2 + z: z \in [-n, n]^2\} \cap \mathbb{Z}^2$, and define

$$T_n = \{b = \{x, y\} \in G: y \notin Y_{<f(b)}(x, \Lambda_n(x, y))\}$$

Note T_n is the same as T except that one only considers paths connecting x to y inside a large box. Also,

$$T_1 \supset T_2 \supset \dots \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} T_n = T$$

Fix x and s and let $n = [\,|x|/4\,] - 2$ (the integer part.) Let E_s be the set of 12 bonds which have at least one endpoint in the closure of the square of form $[i, i+1) \times [j, j+1)$ $(i, j \in \mathbb{Z})$ containing s, with E_{s+x} defined similarly. Then the events $[b \in T_n]$, $b \in E_s$, are independent of the events $[b \in T_n]$, $b \in E_{s+x}$. Since at least one bond of E_s and at least one bond of E_{s+x} are in T_n, it follows that $d(s, T_n)$ is independent of $d(s+x, T_n)$. As $x \to \infty$, so that also $n \to \infty$, we have $d(s, T_n) - d(s, T) \to 0$ and $d(s+x, T_n) - d(s+x, T) \to 0$, both in L^2 uniformly in s. It follows that $\mathrm{cov}(\psi(s+x), \psi(s)) \to 0$ uniformly in s. ■

ACKNOWLEDGMENTS

The research of K.S.A. was supported by NSF grant DMS-9206139, and that of S.A.M. by NSF grant DMS-9310710 and ONR grant N 00014-91-J-1526.

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